

Name: \_\_\_\_\_

Date: \_\_\_\_\_

## Review for Discrete Final, 2011-12

### Voting Review

| Four Methods of Counting Votes  | Four Fairness Criteria |
|---|------------------------|
| 1) Plurality - most 1st place votes wins  | 1)                     |
| 2) Borda: assign points to each choice. Choice w. most pts wins   | 2)                     |
| 3) Pairwise Comparison: head to head w each choice  | 3)                     |
| 4) Plurality w. Elimination: Elim. choice w. fewest 1st place votes. Re-count. Cont. elimination till choice reaches majority | 4)                     |

|                        | 10 | 6 | 5 | 4 | 2 |
|------------------------|----|---|---|---|---|
| 1 <sup>st</sup> choice | A  | B | B | C | D |
| 2 <sup>nd</sup> choice | C  | D | C | A | C |
| 3 <sup>rd</sup> choice | B  | C | A | D | B |
| 4 <sup>th</sup> choice | D  | A | D | B | A |

Majority:  
total votes  
2

round up or add 1 to next whole #.

1) Find the winner under the plurality method.

B has most 1st place votes  
(but doesn't have majority of votes)

Circle any criteria this result violates:

- majority criterion
- independence of irrelevant alternatives criterion
- Condorcet criterion
- monotonicity criterion

none violated

2) Find the winner under the plurality-with-elimination method. (Show every elimination round.)

|   |              |              |   |              |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |  |  |    |   |   |   |   |              |   |   |   |              |              |              |              |   |   |   |   |   |   |   |  |   |    |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
|---|--------------|--------------|---|--------------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|--|--|----|---|---|---|---|--------------|---|---|---|--------------|--------------|--------------|--------------|---|---|---|---|---|---|---|--|---|----|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| <table border="1" style="border-collapse: collapse; text-align: center;"> <tr><td>10</td><td>6</td><td>5</td><td>4</td><td>2</td></tr> <tr><td>A</td><td>B</td><td>B</td><td>C</td><td>D</td></tr> <tr><td>C</td><td>D</td><td>C</td><td>A</td><td>C</td></tr> <tr><td>B</td><td>C</td><td>A</td><td>D</td><td>B</td></tr> <tr><td>D</td><td>A</td><td>D</td><td>B</td><td>A</td></tr> </table> | 10           | 6            | 5 | 4            | 2 | A | B | B | C | D | C | D | C | A | C | B | C | A | D | B | D | A | D | B | A | <p style="font-size: small;">D has fewest 1st place votes</p> <p style="font-size: 2em;">→</p> | <table border="1" style="border-collapse: collapse; text-align: center;"> <tr><td>10</td><td>6</td><td>5</td><td>4</td><td>2</td></tr> <tr><td><del>A</del></td><td>B</td><td>B</td><td>C</td><td><del>C</del></td></tr> <tr><td><del>C</del></td><td><del>C</del></td><td><del>C</del></td><td>A</td><td>B</td></tr> <tr><td>B</td><td>A</td><td>A</td><td>B</td><td>A</td></tr> </table> | 10 | 6 | 5 | 4 | 2 | <del>A</del> | B | B | C | <del>C</del> | <del>C</del> | <del>C</del> | <del>C</del> | A | B | B | A | A | B | A | <p style="font-size: small;">C has fewest 1st place votes</p> <p style="font-size: 2em;">→</p> | <table border="1" style="border-collapse: collapse; text-align: center;"> <tr><td>10</td><td>6</td><td>5</td><td>4</td><td>2</td></tr> <tr><td>A</td><td>B</td><td>B</td><td>A</td><td>B</td></tr> <tr><td>B</td><td>A</td><td>A</td><td>B</td><td>A</td></tr> </table> <p style="text-align: center; font-size: 1.2em;">A has majority</p> | 10 | 6 | 5 | 4 | 2 | A | B | B | A | B | B | A | A | B | A |
| 10  | 6            | 5            | 4 | 2            |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |  |  |    |   |   |   |   |              |   |   |   |              |              |              |              |   |   |   |   |   |   |   |  |   |    |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| A   | B            | B            | C | D            |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |  |  |    |   |   |   |   |              |   |   |   |              |              |              |              |   |   |   |   |   |   |   |  |   |    |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| C   | D            | C            | A | C            |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |  |  |    |   |   |   |   |              |   |   |   |              |              |              |              |   |   |   |   |   |   |   |  |   |    |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| B   | C            | A            | D | B            |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |  |  |    |   |   |   |   |              |   |   |   |              |              |              |              |   |   |   |   |   |   |   |  |   |    |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| D   | A            | D            | B | A            |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |  |  |    |   |   |   |   |              |   |   |   |              |              |              |              |   |   |   |   |   |   |   |  |   |    |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 10  | 6            | 5            | 4 | 2            |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |  |  |    |   |   |   |   |              |   |   |   |              |              |              |              |   |   |   |   |   |   |   |  |   |    |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| <del>A</del>  | B            | B            | C | <del>C</del> |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |  |  |    |   |   |   |   |              |   |   |   |              |              |              |              |   |   |   |   |   |   |   |  |   |    |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| <del>C</del>  | <del>C</del> | <del>C</del> | A | B            |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |  |  |    |   |   |   |   |              |   |   |   |              |              |              |              |   |   |   |   |   |   |   |  |   |    |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| B   | A            | A            | B | A            |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |  |  |    |   |   |   |   |              |   |   |   |              |              |              |              |   |   |   |   |   |   |   |  |   |    |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 10  | 6            | 5            | 4 | 2            |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |  |  |    |   |   |   |   |              |   |   |   |              |              |              |              |   |   |   |   |   |   |   |  |   |    |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| A   | B            | B            | A | B            |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |  |  |    |   |   |   |   |              |   |   |   |              |              |              |              |   |   |   |   |   |   |   |  |   |    |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| B   | A            | A            | B | A            |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |  |  |    |   |   |   |   |              |   |   |   |              |              |              |              |   |   |   |   |   |   |   |  |   |    |   |   |   |   |   |   |   |   |   |   |   |   |   |   |

Circle any criteria this result violates:

- majority criterion
- independence of irrelevant alternatives criterion
- Condorcet criterion
- monotonicity criterion

A wins

maj = 13

|                        | 8              | 7              | 6              | 2              | 1              |
|------------------------|----------------|----------------|----------------|----------------|----------------|
| 1 <sup>st</sup> choice | A <sub>5</sub> | D <sub>5</sub> | D <sub>5</sub> | C <sub>3</sub> | E <sub>5</sub> |
| 2 <sup>nd</sup> choice | B <sub>4</sub> | B <sub>4</sub> | B <sub>4</sub> | A <sub>4</sub> | A <sub>4</sub> |
| 3 <sup>rd</sup> choice | C <sub>3</sub> | A <sub>3</sub> | E <sub>3</sub> | B <sub>3</sub> | D <sub>3</sub> |
| 4 <sup>th</sup> choice | D <sub>2</sub> | C <sub>2</sub> | C <sub>2</sub> | D <sub>2</sub> | B <sub>2</sub> |
| 5 <sup>th</sup> choice | E <sub>1</sub> | E <sub>1</sub> | A <sub>1</sub> | E <sub>1</sub> | C <sub>1</sub> |

3) Find the winner under the Borda method.

$$\begin{aligned}
 A &= 8(5) + 7(3) + 6(1) + 2(4) + 1(4) = 79 \\
 B &= 8(4) + 7(4) + 6(4) + 2(3) + 1(2) = 92 \\
 C &= 8(3) + 7(2) + 6(2) + 2(5) + 1(1) = 61 \\
 D &= 8(2) + 7(5) + 6(5) + 2(2) + 1(3) = 88 \\
 E &= 8(1) + 7(1) + 6(3) + 2(1) + 1(5) = 40
 \end{aligned}$$

B wins

However, this violates majority criterion because D already has majority of 1st place votes.

Circle any criteria this result violates:

- majority criterion
- Condorcet criterion
- monotonicity criterion
- independence of irrelevant alternatives criterion

This is a 2-part story. Keep eye out for monotonicity or independence violations.

maj: 27

|                        | 27             | 24             | 2              |
|------------------------|----------------|----------------|----------------|
| 1 <sup>st</sup> choice | A <sub>3</sub> | P <sub>3</sub> | C <sub>3</sub> |
| 2 <sup>nd</sup> choice | C <sub>2</sub> | C <sub>2</sub> | P <sub>2</sub> |
| 3 <sup>rd</sup> choice | P <sub>1</sub> | A <sub>1</sub> | A <sub>1</sub> |

4) Consider the straw vote election, to be determined by the Borda count method, given the above schedule.

$$\begin{aligned}
 A &= 27(3) + 24(1) + 2(1) = 107 \\
 C &= 27(2) + 24(2) + 2(2) = 106 \\
 P &= 27(1) + 24(3) + 2(3) = 105
 \end{aligned}$$

A wins

P, upset by his loss, drops out before the real election could be counted. Who wins the official election?

|   | 27             | 24             | 2              |
|---|----------------|----------------|----------------|
| A | A <sub>2</sub> | C <sub>2</sub> | C <sub>2</sub> |
| C | C <sub>1</sub> | A <sub>1</sub> | A <sub>1</sub> |

$$\begin{aligned}
 A &= 27(2) + 24(1) + 2(1) = 80 \quad \underline{\text{A wins}} \\
 C &= 27(1) + 24(2) + 2(2)
 \end{aligned}$$

If A had lost, because P

Circle any criteria this result violates:

- majority criterion
- Condorcet criterion
- monotonicity criterion
- independence of irrelevant alternatives criterion

dropped out, it would have violated

maj: 9

|                        |   |   |   |   |
|------------------------|---|---|---|---|
|                        | 6 | 5 | 4 | 2 |
| 1 <sup>st</sup> choice | A | B | C | C |
| 2 <sup>nd</sup> choice | B | A | B | A |
| 3 <sup>rd</sup> choice | C | C | A | B |

5) A straw vote election is held. Find the winner under the plurality-with-elimination method. (Show every elimination round.)

B has fewest 1st place votes

|   |   |   |   |   |
|---|---|---|---|---|
|   | 6 | 5 | 4 | 2 |
| A | A | A | C | C |
| B | C | C | A | A |

A wins!

In the official election, the two voters change their votes from C, A, B to A, C, B. Find the winner under the plurality-with-elimination method. (Show every elimination round.)

changed votes →

|   |   |   |   |   |
|---|---|---|---|---|
|   | 6 | 5 | 4 | 2 |
| A | A | B | C | A |
| B | B | A | B | C |
| C | C | A | A | B |

C has fewest 1st place votes

|   |   |   |   |   |
|---|---|---|---|---|
|   | 6 | 5 | 4 | 2 |
| A | A | B | B | A |
| B | B | A | A | B |

B wins

Circle any criteria this result violates:

- majority criterion
- Condorcet criterion
- monotonicity criterion
- independence of irrelevant alternatives criterion

Violates monotonicity: A was going to win. Votes changed in favor of A, and A lost.

6) Rank the candidates using the recursive plurality method.

|                        |    |    |   |   |   |
|------------------------|----|----|---|---|---|
|                        | 14 | 10 | 8 | 4 | 1 |
| 1 <sup>st</sup> choice | A  | C  | D | B | C |
| 2 <sup>nd</sup> choice | B  | B  | C | D | D |
| 3 <sup>rd</sup> choice | C  | D  | B | C | B |
| 4 <sup>th</sup> choice | D  | A  | A | A | A |

Winner: \_\_\_\_\_

2<sup>nd</sup> place: \_\_\_\_\_

3<sup>rd</sup> place: \_\_\_\_\_

4<sup>th</sup> place: \_\_\_\_\_

# Weighted Voting Review

|   | Banzhaf<br>[8: 4, 3, 2, 1]   | Shapley-Shubik<br>[8: 4, 3, 2, 1]   |
|---|--|---|
| 1) How are coalitions created?                            | By groups  | Over time - order matters   |
| 2) List all possible coalitions.                          | $\begin{matrix} P_1 & P_1 P_2 & P_3 P_4 & P_2 P_3 P_4 \\ P_2 & P_1 P_3 & P_1 P_2 P_3 & P_1 P_2 P_3 P_4 \\ P_3 & P_1 P_4 & P_1 P_2 P_4 & \\ P_4 & P_2 P_3 & P_1 P_3 P_4 & \end{matrix}$ | $\begin{matrix} P_1 \\ P_1 P_2 \\ P_1 P_3 \\ P_1 P_4 \\ P_1 P_2 P_3 \\ P_1 P_2 P_4 \\ P_1 P_3 P_4 \\ P_1 P_2 P_3 P_4 \end{matrix}$  |
| 3) Formula for number of coalitions.                      | $2^n - 1$ ← n is # of players  | $n!$  |
| 4) What are the most influential players called?          | critical players   | pivotal players   |
| 5) How do you find those players?                         | If a winning coalition becomes a losing coalition because player drops out   | add player weights L → R<br>The player that makes coalition win is pivotal player   |
| 6) What is the power distribution of each of the players? | $P_1: \underline{\frac{3}{8}}$ $P_4: \underline{\frac{1}{8}}$<br><br>$P_2: \underline{\frac{3}{8}}$<br><br>$P_3: \underline{\frac{1}{8}}$  | $P_1: \underline{\frac{10}{24} = \frac{5}{12}}$ $P_4: \underline{\frac{2}{24} = \frac{1}{12}}$<br><br>$P_2: \underline{\frac{10}{24} = \frac{5}{12}}$<br><br>$P_3: \underline{\frac{2}{24} = \frac{1}{12}}$ |

1) Consider the weighted voting system [47: 10,9,9,5,4,4,3,2,2]

- How many players are there? 9
- What is the total number (weight) of votes? 48
- What is the quota in this system? 47

2) Consider the weighted voting system [q: 7,5,3,1,1]

- What is the smallest value that the quota q can take? (4)
- What is the largest value that the quota q can take? (17)
- What is the value of the quota if at least two-thirds of the votes are required to pass a motion?

total votes 17  
 smallest  $\frac{17}{2} = 13.5 = 14$

$\rightarrow \frac{2}{3}(17) = 11.3 = 12$

Handwritten lists of permutations for the weighted voting system [q: 7,5,3,1,1].

- Column 1:  $P_1 P_2 P_3 P_4$ ,  $P_1 P_2 P_4 P_3$ ,  $P_1 P_3 P_2 P_4$ ,  $P_1 P_3 P_4 P_2$ ,  $P_1 P_4 P_2 P_3$ ,  $P_1 P_4 P_3 P_2$
- Column 2:  $P_2 P_1 P_3 P_4$ ,  $P_2 P_1 P_4 P_3$ ,  $P_2 P_3 P_1 P_4$ ,  $P_2 P_3 P_4 P_1$ ,  $P_2 P_4 P_1 P_3$ ,  $P_2 P_4 P_3 P_1$
- Column 3:  $P_3 P_1 P_2 P_4$ ,  $P_3 P_1 P_4 P_2$ ,  $P_3 P_2 P_1 P_4$ ,  $P_3 P_2 P_4 P_1$ ,  $P_3 P_4 P_1 P_2$ ,  $P_3 P_4 P_2 P_1$
- Column 4:  $P_4 P_1 P_2 P_3$ ,  $P_4 P_1 P_3 P_2$ ,  $P_4 P_2 P_1 P_3$ ,  $P_4 P_2 P_3 P_1$ ,  $P_4 P_3 P_1 P_2$ ,  $P_4 P_3 P_2 P_1$

3) Consider the weighted voting system [13: 13, 6, 4, 2]

- g. Identify the dictators, if any.  $P_1$
- h. Identify players with veto power, if any. —
- i. Identify dummies, if any.  $P_2, P_3, P_4$

4) Consider the weighted voting system [95: 65, 35, 30, 25].

- a. Identify the dictators, if any. —
- b. Identify players with veto power, if any.  $P_1$
- c. Identify dummies, if any. —

5) Consider the weighted voting system [q: 5, 3, 2, 2, 1, 1].

total votes = 14

- a. What is largest possible value of  $q$ ? 14
- b. What is the smallest possible value of  $q$ ? 8
- c. If no player is to have veto power, what is the largest possible value of  $q$ ? 9

6) Consider the weighted voting system [15: 6, 5, 3, 2, 1, 1].

- a. How many possible coalitions are there in this system?
- b. How many sequential coalitions are there in this system?

$$2^6 - 1 = 63$$
$$6! = 720$$

7) Consider the weighted voting system [15: 11, 7, 5, 2]

- a. What is the weight of the coalition  $\{P_1, P_2, P_4\}$  20
- b. In the coalition  $\{P_1, P_2, P_4\}$  which players are critical?  $P_1, P_2$

8) Consider the weighted voting system [15: 13, 9, 5, 2]. In the sequential coalition  $\langle P_1, P_4, P_2, P_3 \rangle$  which player is pivotal?

$P_4$

9) Find the Banzhaf power distribution for the system [8: 5, 3, 2].

$$P_1 = 50\%$$
$$P_2 = 50\%$$
$$P_3 = 0\%$$

10) Find the Shapley-Shubik power distribution for the system [18: 19, 7, 5, 3, 2, 1].

$P_1$  is a dictator  $P_1 = 100\%$

11) Find the Banzhaf power distribution for the system [37: 19, 7, 5, 3, 2, 1].

total votes = 37, the quota.  
ALL PLAYERS HAVE EQUAL POWER

12) In the weighted voting system [q: 8, 5, 4, 1], if every voter has veto power what is  $q$ ?

$$q = 18$$

13) In the weighted voting system  $[q: 6, 5, 4, 3, 2, 1]$ , what is the largest value of  $q$  so that no voter has veto power?

14) A weighted voting system has a Shapley-Shubik power index of  $(\frac{1}{2}, \frac{1}{3}, \frac{1}{10}, x, \frac{1}{120})$ . Find  $x$ .

$$1 - \left( \frac{1}{2} + \frac{1}{3} + \frac{1}{10} + \frac{1}{120} \right) = x$$

$\frac{7}{120}$

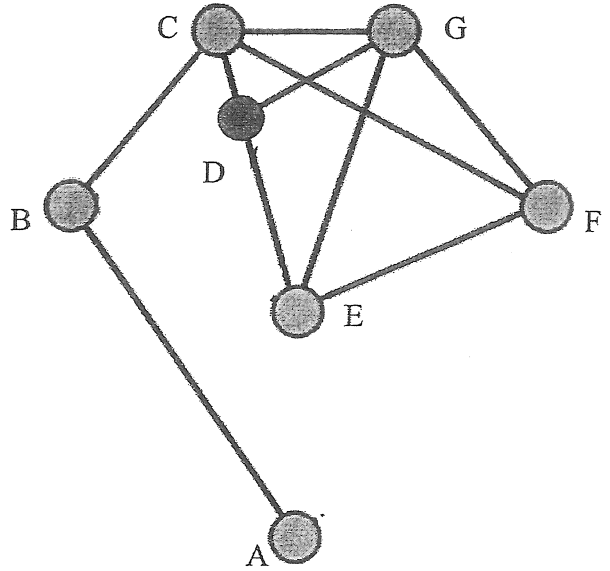
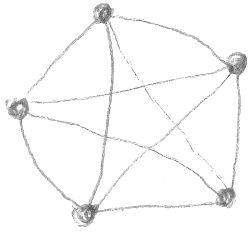
15) A weighted voting system has a Banzhaf power index of  $(31\%, 24\%, 20\%, 18\%, x)$ . Find  $x$ .

$$100\% - 31\% - 24\% - 20\% - 18\% = x$$

$$x = 7$$

# Euler Review

1) Draw a connected graph with five vertices, each of degree 4 with both multiple edges and loops.



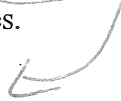
2) The following questions refer to

a. Find a circuit of length 3 passing through vertex E.

b. Find a path of length 6 from A to D.

*BC, CB*

c. Find the bridge/bridges.



d. Which edge could be added so that the resulting graph has no bridges?

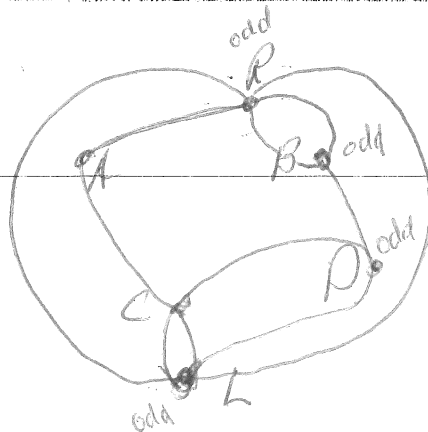
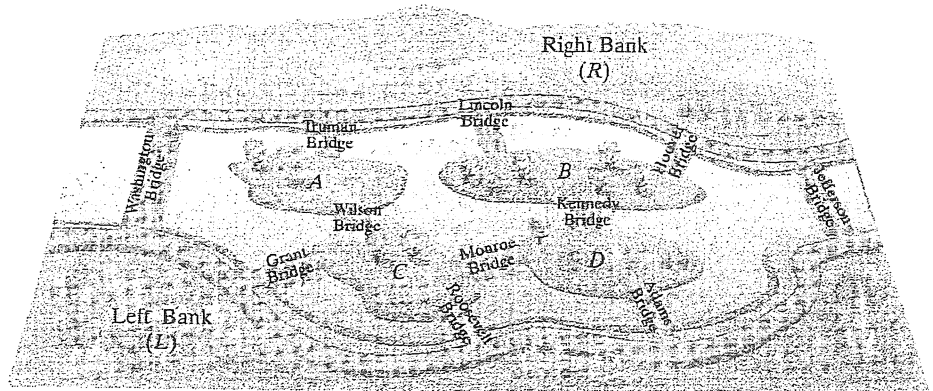
*AF*

3) Draw a graph based on the figure.

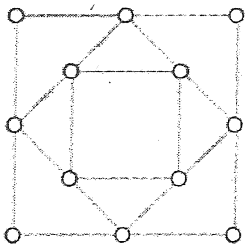
Is it possible to travel every bridge once and only once? *no*

If yes, trace the graph. If no, explain why.

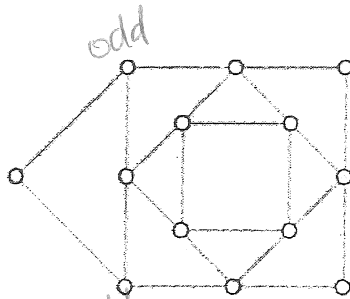
*there are 4 odd vertices*



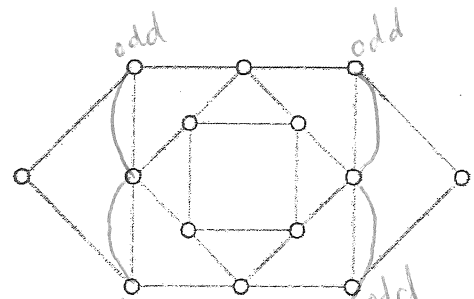
4) Determine whether each graph has an Euler circuit, Euler path, or neither.



(a)



(b)



(c)

Figure A:

Euler circuit? ✓  
 Euler path?  
 Neither

Figure B:

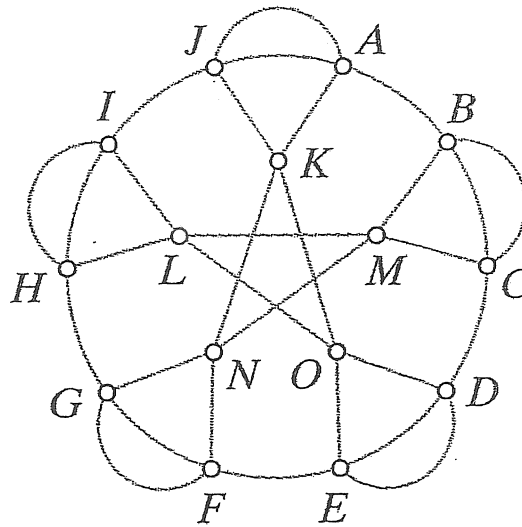
Euler circuit?  
 Euler path? ✓  
 Neither

Figure C:

Euler circuit?  
 Euler path?  
 Neither ✓

5) If you chose neither on any of the above graphs, Eulerize the graph.

6) Is it possible to find an Euler circuit, Euler path or neither on the following graph? Trace the circuit or path, if possible.



yes:  
 Euler circuit